$$\int \frac{2x^2+5}{x^3-x^2-x+1}\,dx = \int \frac{2x^2+5}{(x-1)^2(x+1)}\,dx \quad \text{(factor)}$$

$$= \int \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\,dx$$

System

$A(x-1)(x+1) = Ax^2 - A$

$B(x+1) = Bx + B$

$C(x-1)^2 = Cx^2 - 2Cx + C$

$x^2: \; A + C = 2$

$x^1: \; B - 2C = 0$

$x^0: \; -A + B + C = 5$

$A = \frac{1}{4} \quad B = \frac{7}{2} \quad C = \frac{7}{4}$

$$= \frac{1}{4}\int \frac{dx}{x-1} + \frac{7}{2}\int \frac{dx}{(x-1)^2} + \frac{7}{4}\int \frac{dx}{x+1}$$

$$= \frac{1}{4}\ln|x-1| - \frac{7}{2}\,\frac{1}{x-1} + \frac{7}{4}\ln|x+1| + C$$

$$\int \frac{-9x^2+17x+32}{(x-4)(x^2+36)}\,dx = \int \frac{A}{x-4} + \frac{Bx+C}{x^2+36}\,dx$$

System

$A(x^2+36) = Ax^2+36A$

$(Bx+C)(x-4) = Bx^2-4Bx+Cx-4C$

$x^2:\ A+B=-9$

$x^1:\ -4B+C=17$

$x^0:\ 36A-4C=32$

$A=\frac{-11}{13}\quad B=-\frac{106}{13}\quad C=\frac{-203}{13}$

$$= -\frac{11}{13}\int\frac{dx}{x-4} + -\frac{1}{13}\int\frac{106x+203}{x^2+36}\,dx$$

let $u=x^2+36$

$du=2x\,dx$

$$= -\frac{11}{13}\ln|x-4| - \frac{53}{13}\int\frac{2x\,dx}{x^2+36} + -\frac{203}{13}\int\frac{1}{x^2+36}\,d$$

$$= -\frac{11}{13}\ln|x-4| - \frac{53}{13}\ln(x^2+36) - \frac{203}{13}\cdot\frac{1}{6}\arctan\left(\frac{x}{6}\right)+$$

$$= -\frac{11}{13}\ln|x-4| - \frac{53}{13}\ln(x^2+36) - \frac{203}{84}\arctan\left(\frac{x}{6}\right)+C$$

$$\int x^3\sqrt{9-x^2}\,dx$$

let $x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$=\int (3\sin\theta)^3\sqrt{9-(3\sin\theta)^2}\;3\cos\theta\,d\theta$$

$$=\int 243\sin^3\theta\cos^2\theta\,d\theta$$

$u=\cos\theta$

$du=-\sin\theta\,d\theta$

$$=-\int 243(1-u^2)u^2\,du$$

$$=-243\int u^2-u^4\,du$$

$$=-243\left(\frac{u^3}{3}-\frac{u^5}{5}\right)+C$$

$$=-243\left(\frac{\cos^3\theta}{3}-\frac{\cos^5\theta}{5}\right)+C$$

$x=3\sin\theta \Leftrightarrow \sin\theta=\frac{x}{3}$

$\cos\theta=\sqrt{1-\sin^2\theta}$

$=\sqrt{1-\frac{x^2}{9}}$

$=\sqrt{\frac{9-x^2}{9}}$

$=\frac{1}{3}\sqrt{9-x^2}$

$$=-243\left(\frac{\left(\frac{1}{3}\sqrt{9-x^2}\right)^3}{3}-\frac{\left(\frac{1}{3}\sqrt{9-x^2}\right)^5}{5}\right)+C$$

$$=-243\left(\frac{\left(\sqrt{9-x^2}\right)^3}{81}-\frac{\left(\sqrt{9-x^2}\right)^5}{3^5\cdot 5}\right)+C$$

$$\left(3\sqrt{9-x^2}\right)^3+\frac{\left(\sqrt{9-x^2}\right)^5}{5}+C$$

$$\int \frac{dx}{x^2\sqrt{16x^2-9}}$$

let $x = \frac{3}{4}\sec\theta$

$dx = \frac{3}{4}\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\frac{3}{4}\sec\theta\tan\theta\, d\theta}{\left(\frac{3}{4}\sec\theta\right)^2\sqrt{16\left(\frac{3}{4}\sec\theta\right)^2-9}}$$

$$= \int \frac{\cancel{\frac{3}{4}}\,\cancel{\sec\theta}\tan\theta\, d\theta}{\left(\frac{3}{4}\right)^{\cancel{2}}\sec^{\cancel{2}}\theta\sqrt{9\sec^2\theta-9}}$$

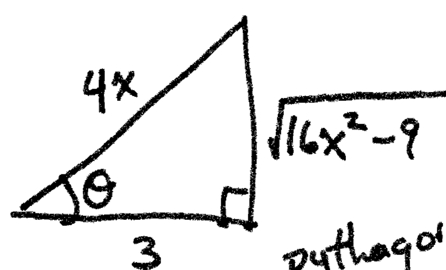
$$= \int \frac{\cancel{\tan\theta}\, d\theta}{\frac{3}{4}\sec\theta\cdot 3\cancel{\tan\theta}}$$

$$= \frac{4}{9}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{4}{9}\int \cos\theta\, d\theta$$

$$= \frac{4}{9}\sin\theta + C$$

$x = \frac{3}{4}\sec\theta \Longleftrightarrow \sec\theta = \frac{4x}{3}\left(\frac{H}{A}\right)$



Pythagorean theorem

so $\sin\theta = \frac{\sqrt{16x^2-9}}{4x}$

$$= \frac{\cancel{4}}{9}\frac{\sqrt{16x^2-9}}{\cancel{4}x} + C$$

$$= \frac{\sqrt{16x^2-9}}{9x} + C$$

$$\int \frac{x^2\,dx}{\sqrt{4x-x^2}} = \int \frac{x^2\,dx}{\sqrt{-(x^2-4x)}} = \int \frac{x^2\,dx}{\sqrt{-(x^2-4x+4-4)}}$$

$$= \int \frac{x^2\,dx}{\sqrt{-((x-2)^2-4}} = \int \frac{x^2\,dx}{\sqrt{4-(x-2)^2}}$$

$x-2 = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$$= \int \frac{(2\sin\theta+2)^2(2\cos\theta)\,d\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{(2\sin\theta+2)^2\cancel{(2\cos\theta)}\,d\theta}{\cancel{2\cos\theta}}$$

$$= \int 4\sin^2\theta + 8\sin\theta + 4\,d\theta$$

$$= \int \cancel{4}^{2}\left(\tfrac{1}{\cancel{2}}(1-\cos(2\theta))\right) + 8\sin\theta + 4\,d\theta$$

add 2 and 4 for constant term

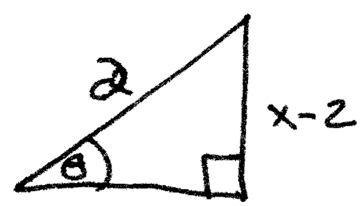
$$= \int 6 - 2\cos(2\theta) + 8\sin\theta\,d\theta$$

$$= 6\theta - \sin(2\theta) - 8\cos\theta + C$$

$\sin\theta = \frac{x-2}{2}$

double angle identity

$$= 6\theta - 2\sin\theta\cos\theta - 8\cos\theta + C$$



$$= 6\arcsin\left(\frac{x-2}{2}\right) - 2\left(\frac{x-2}{2}\right)\left(\frac{\sqrt{4x-x^2}}{2}\right) - 8\left(\frac{\sqrt{4x-x^2}}{2}\right) + C$$

$\sqrt{4-(x-2)^2} = \sqrt{4x-x^2}$

$\cos\theta = \frac{\sqrt{4x-x^2}}{2}$

$$= 6\arcsin\left(\frac{x-2}{2}\right) - (x-2)\left(\frac{\sqrt{4x-x^2}}{2}\right) - 4\sqrt{4x-x^2} + C$$

$$\int_0^1 x^3 \sqrt{x^2+1}\, dx$$

let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$= \int_0^{\frac{\pi}{4}} \tan^3\theta \sqrt{\tan^2\theta + 1}\, \sec^2\theta \, d\theta$$

$1 = \tan\theta \Rightarrow \theta = \frac{\pi}{4}$

$0 = \tan\theta \Rightarrow \theta = 0$

$$= \int_0^{\frac{\pi}{4}} \tan^3\theta \sec^3\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^2\theta \sec^2\theta \tan\theta \sec\theta \, d\theta$$

$\tan^2\theta = \sec^2\theta - 1$

let ~~u = tan~~ $u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$$= \int_1^{\sqrt{2}} (u^2 - 1) u^2 \, du$$

$u = \sec 0 = 1$

$u = \sec\frac{\pi}{4} = \sqrt{2}$

$$= \int_1^{\sqrt{2}} u^4 - u^2 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} * \Big]_1^{\sqrt{2}}$$

$$= \frac{(\sqrt{2})^5}{5} - \frac{(\sqrt{2})^3}{3} - \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= \frac{4\sqrt{2} - 1}{5} + \frac{1 - 2\sqrt{2}}{3}$$

≈ 0.3218951416