1. Trigonometric Identities

Prove the following identities.

1. $\sin(\theta)\sec(\theta) = \tan(\theta)$
2. $\cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$
3. $\dfrac{\csc(x)}{\sec(x)} = \cot(x)$
4. $(1+\cos(\beta))(1-\cos(\beta)) = \sin^2(\beta)$
5. $\cot(w)\sin(w) = \cos(w)$
6. $\cos^2(x)(\sec^2(x) - 1) = \sin^2(x)$
7. $\sec(\alpha) - \cos(\alpha) = \sin(\alpha)\tan(\alpha)$
8. $\dfrac{\cos(x)}{1-\sin(x)} = \dfrac{1+\sin(x)}{\cos(x)}$
9. $(\tan(\theta) + \cot(\theta))\tan(\theta) = \sec^2(\theta)$
10. $\dfrac{\sin(t)}{\csc(t)} + \dfrac{\cos(t)}{\sec(t)} = 1$
11. $\sin(t)(\csc(t) - \sin(t)) = \cos^2(t)$
12. $\dfrac{\csc^2(\theta)}{1+\tan^2(\theta)} = \cot^2(\theta)$
13. $\dfrac{\tan(t)+\cos(t)}{\sin(t)} = \sec(t) + \cot(t)$

2. Trigonometric Equations

Algebraically solve the following equations. Exact solutions only!

1. $2\sin(x) = \sqrt{3}$
2. $\sin(2\alpha) + 1 = 0, \qquad 0 \le \alpha < 2\pi$
3. $\cos(2x) = \dfrac{1}{2}, \qquad 0 \le x < 2\pi$
4. $(4\sin^2(p) - 2)(4\cos^2(p) - 3) = 0$
5. $2\sin^2(\theta) - \cos(\theta) - 1 = 0$
6. $\sin(2x) + \cos(x) = 0, \qquad 0 \le x < 2\pi$
7. $2\sin^2(t) + \cos^2(t) = 1, \qquad 0 \le t < 2\pi$

8. $4\sin^2(x)\tan(x) - \tan(x) = 0, \qquad 0 \le x < 2\pi$

9. $\sin(2\omega) - 3\sin(\omega) = 0, \qquad 0 \le \omega < 2\pi$

10. $2\cos^2(x) - \sin(x) = 1, \qquad 0 \le x < 2\pi$

11. $3\sin^2(\beta) - 2\cos(\beta) = 2, \qquad 0 \le \beta < 2\pi$

12. $2\cos^2(t) + 7\sin(t) = 5, \qquad 0 \le t < 2\pi$

3. Solutions for Identities, Section 1.

1. $\sin(\theta)\sec(\theta) = \tan(\theta)$

Solution:

$$\begin{aligned} \sin(\theta)\sec(\theta) &= \sin(\theta) \cdot \frac{1}{\cos(\theta)} \\ &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \tan(\theta) \qquad \checkmark \end{aligned}$$

2. $\cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$

Solution:

$$\begin{aligned} \cos^2(t) - \sin^2(t) &= \cos^2(t) - (1 - \cos^2(t)) \\ &= \cos^2(t) - 1 + \cos^2(t) \\ &= 2\cos^2(t) - 1 \qquad \checkmark \end{aligned}$$

3. $\dfrac{\csc(x)}{\sec(x)} = \cot(x)$

Solution:

$$\begin{aligned} \frac{\csc(x)}{\sec(x)} &= \frac{\frac{1}{\sin(x)}}{\frac{1}{\cos(x)}} \\ &= \frac{1}{\sin(x)} \cdot \frac{\cos(x)}{1} \\ &= \frac{\cos(x)}{\sin(x)} \\ &= \cot(x) \qquad \checkmark \end{aligned}$$

4. $(1 + \cos(\beta))(1 - \cos(\beta)) = \sin^2(\beta)$

Solution:

$$\begin{aligned} (1 + \cos(\beta))(1 - \cos(\beta)) &= 1 - \cos^2(\beta) \\ &= \sin^2(\beta) \qquad \checkmark \end{aligned}$$

5. $\cot(w)\sin(w) = \cos(w)$

Solution:

$$\begin{aligned}\cot(w)\sin(w) &= \frac{\cos(w)}{\sin(w)} \cdot \sin(w) \\ &= \cos(w) \qquad \checkmark\end{aligned}$$

6. $\cos^2(x)(\sec^2(x) - 1) = \sin^2(x)$

Solution:

$$\begin{aligned}\cos^2(x)(\sec^2(x) - 1) &= \cos^2(x)\left(\frac{1}{\cos^2(x)} - 1\right) \\ &= \frac{\cos^2(x)}{\cos^2(x)} - \cos^2(x) \\ &= 1 - \cos^2(x) \\ &= \sin^2(x) \qquad \checkmark\end{aligned}$$

7. $\sec(\alpha) - \cos(\alpha) = \sin(\alpha)\tan(\alpha)$

Solution:

$$\begin{aligned}\sec(\alpha) - \cos(\alpha) &= \frac{1}{\cos(\alpha)} - \cos(\alpha) \\ &= \frac{1}{\cos(\alpha)} - \cos(\alpha) \cdot \frac{\cos(\alpha)}{\cos(\alpha)} \\ &= \frac{1}{\cos(\alpha)} - \frac{\cos^2(\alpha)}{\cos(\alpha)} \\ &= \frac{1 - \cos^2(\alpha)}{\cos(\alpha)} \\ &= \frac{\sin^2(\alpha)}{\cos(\alpha)} \\ &= \sin(\alpha) \cdot \frac{\sin(\alpha)}{\cos(\alpha)} \\ &= \sin(\alpha)\tan(\alpha) \qquad \checkmark\end{aligned}$$

8. $\dfrac{\cos(x)}{1 - \sin(x)} = \dfrac{1 + \sin(x)}{\cos(x)}$

Solution:

$$\begin{aligned}\frac{\cos(x)}{1 - \sin(x)} &= \frac{\cos(x)}{1 - \sin(x)} \cdot \frac{1 + \sin(x)}{1 + \sin(x)} \\ &= \frac{\cos(x)(1 + \sin(x))}{(1 - \sin(x))(1 + \sin(x))} \\ &= \frac{\cos(x)(1 + \sin(x))}{1 - \sin^2(x)} \\ &= \frac{\cos(x)(1 + \sin(x))}{\cos^2(x)} \\ &= \frac{1 + \sin(x)}{\cos(x)} \qquad \checkmark\end{aligned}$$

9. $(\tan(\theta) + \cot(\theta))\tan(\theta) = \sec^2(\theta)$

Solution:

$$\begin{aligned}
(\tan(\theta) + \cot(\theta))\tan(\theta) &= \left(\frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}\right) \cdot \frac{\sin(\theta)}{\cos(\theta)} \\
&= \frac{\sin^2(\theta)}{\cos^2(\theta)} + 1 \\
&= \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} \\
&= \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)} \\
&= \frac{1}{\cos^2(\theta)} \\
&= \sec^2(\theta) \qquad \surd
\end{aligned}$$

10. $\dfrac{\sin(t)}{\csc(t)} + \dfrac{\cos(t)}{\sec(t)} = 1$

Solution:

$$\begin{aligned}
\frac{\sin(t)}{\csc(t)} + \frac{\cos(t)}{\sec(t)} &= \frac{\sin(t)}{\frac{1}{\sin(t)}} + \frac{\cos(t)}{\frac{1}{\cos(t)}} \\
&= \frac{\frac{\sin(t)}{1}}{\frac{1}{\sin(t)}} + \frac{\frac{\cos(t)}{1}}{\frac{1}{\cos(t)}} \\
&= \frac{\sin(t)}{1} \cdot \frac{\sin(t)}{1} + \frac{\cos(t)}{1} \cdot \frac{\cos(t)}{1} \\
&= \sin^2(t) + \cos^2(t) \\
&= 1 \qquad \surd
\end{aligned}$$

11. $\sin(t)(\csc(t) - \sin(t)) = \cos^2(t)$

Solution:

$$\begin{aligned}
\sin(t)(\csc(t) - \sin(t)) &= \sin(t)\left(\frac{1}{\sin(t)} - \sin(t)\right) \\
&= 1 - \sin^2(t) \\
&= \cos^2(t) \qquad \surd
\end{aligned}$$

12. $\dfrac{\csc^2(\theta)}{1+\tan^2(\theta)} = \cot^2(\theta)$

Solution:

$$\begin{aligned}
\frac{\csc^2(\theta)}{1+\tan^2(\theta)} &= \frac{\csc^2(\theta)}{1+\dfrac{\sin^2(\theta)}{\cos^2(\theta)}} \\
&= \frac{\csc^2(\theta)}{\dfrac{\cos^2(\theta)}{\cos^2(\theta)}+\dfrac{\sin^2(\theta)}{\cos^2(\theta)}} \\
&= \frac{\csc^2(\theta)}{\dfrac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)}} \\
&= \frac{\csc^2(\theta)}{\dfrac{1}{\cos^2(\theta)}} \\
&= \csc^2(\theta)\cdot\cos^2(\theta) \\
&= \frac{1}{\sin^2(\theta)}\cdot\cos^2(\theta) \\
&= \frac{\cos^2(\theta)}{\sin^2(\theta)} \\
&= \cot^2(\theta) \qquad \checkmark
\end{aligned}$$

13. $\dfrac{\tan(t)+\cos(t)}{\sin(t)} = \sec(t)+\cot(t)$

Solution:

$$\begin{aligned}
\frac{\tan(t)+\cos(t)}{\sin(t)} &= \frac{\dfrac{\sin(t)}{\cos(t)}+\cos(t)}{\sin(t)} \\
&= \left(\frac{\sin(t)}{\cos(t)}+\cos(t)\right)\cdot\frac{1}{\sin(t)} \\
&= \frac{1}{\cos(t)}+\frac{\cos(t)}{\sin(t)} \\
&= \sec(t)+\cot(t) \qquad \checkmark
\end{aligned}$$

4. Solutions for Equations, Section 2.

Throughout this section, k will stand for any integer.

1. $2\sin(x) = \sqrt{3}$

Solution:

$$\sin(x) = \frac{\sqrt{3}}{2}$$

$$x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2k\pi, \quad \pi - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2k\pi$$

$$x = \frac{\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi$$

2. $\sin(2\alpha) + 1 = 0, \qquad 0 \le \alpha < 2\pi$

Solution:

$$\begin{gathered}
\sin(2\alpha) + 1 = 0 \\
\sin(2\alpha) = -1 \\
2\alpha = \sin^{-1}(-1) + 2\pi, \quad \sin^{-1}(-1) + 4\pi \\
\big(\text{ note that since } 0 \le \alpha < 2\pi, \text{ we get } 0 \le 2\alpha < 4\pi\big) \\
2\alpha = -\frac{\pi}{2} + 2\pi, -\frac{\pi}{2} + 4\pi \\
2\alpha = \frac{3\pi}{2}, \frac{7\pi}{2} \\
\alpha = \frac{3\pi}{4}, \frac{7\pi}{4}
\end{gathered}$$

3. $\cos(2x) = \frac{1}{2}, \qquad 0 \le x < 2\pi$

Solution:

$$\begin{gathered}
\text{note that since } 0 \le x < 2\pi, \text{ we get } 0 \le 2x < 4\pi \\
2x = \cos^{-1}\left(\frac{1}{2}\right), 2\pi - \cos^{-1}\left(\frac{1}{2}\right), 2\pi + \cos^{-1}\left(\frac{1}{2}\right), 4\pi - \cos^{-1}\left(\frac{1}{2}\right) \\
2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\
x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}
\end{gathered}$$

4. $(4\sin^2(p) - 2)(4\cos^2(p) - 3) = 0$

Solution:

$$\begin{gathered}
\sin^2(p) = \frac{1}{2}, \qquad \cos^2(p) = \frac{3}{4} \\
\sin(p) = \pm\frac{\sqrt{2}}{2}, \qquad \cos(p) = \pm\frac{\sqrt{3}}{2} \\
p = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + 2k\pi, \pi - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + 2k\pi, 2\pi + \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) + 2k\pi, \pi - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) + 2k\pi \\
p = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2k\pi, 2\pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2k\pi, \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + 2k\pi, 2\pi - \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + 2k\pi \\
p = \frac{\pi}{4} + 2k\pi, \frac{3\pi}{4} + 2k\pi, \frac{7\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi, \frac{\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi \\
p = \frac{\pi}{4} + k\pi, \frac{3\pi}{4} + k\pi, \frac{\pi}{6} + k\pi, \frac{5\pi}{6} + k\pi
\end{gathered}$$

5. $2\sin^2(\theta) - \cos(\theta) - 1 = 0$

Solution:

$$
\begin{gathered}
2(1-\cos^2(\theta)) - \cos(\theta) - 1 = 0 \\
2 - 2\cos^2(\theta) - \cos(\theta) - 1 = 0 \\
-2\cos^2(\theta) - \cos(\theta) + 1 = 0 \\
2\cos^2(\theta) + \cos(\theta) - 1 = 0 \\
(2\cos(\theta) - 1)(\cos(\theta) + 1) = 0 \\
2\cos(\theta) - 1 = 0, \qquad \cos(\theta) + 1 = 0 \\
\cos(\theta) = \frac{1}{2}. \qquad \cos(\theta) = -1 \\
\theta = \cos^{-1}\left(\frac{1}{2}\right) + 2k\pi, 2\pi - \cos^{-1}\left(\frac{1}{2}\right) + 2k\pi, \qquad \theta = \cos^{-1}(-1) + 2k\pi \\
\theta = \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi, (2k+1)\pi
\end{gathered}
$$

6. $\sin(2x) + \cos(x) = 0, \qquad 0 \le x < 2\pi$

Solution:

$$
\begin{gathered}
2\sin(x)\cos(x) + \cos(x) = 0 \\
\cos(x)(2\sin(x) + 1) = 0 \\
\cos(x) = 0, \qquad 2\sin(x) + 1 = 0 \\
x = \cos^{-1}(0), 2\pi - \cos^{-1}(0), \qquad \sin(x) = -\frac{1}{2} \\
x = \frac{\pi}{2}, \frac{3\pi}{2}, \qquad x = 2\pi + \sin^{-1}\left(-\frac{1}{2}\right), \pi - \sin^{-1}\left(-\frac{1}{2}\right) \\
x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{11\pi}{6}, \frac{7\pi}{6}
\end{gathered}
$$

7. $2\sin^2(t) + \cos^2(t) = 1, \qquad 0 \le t < 2\pi$

Solution:

$$
\begin{gathered}
\sin^2(t) + \sin^2(t) + \cos^2(t) = 1 \\
\sin^2(t) + 1 = 1 \\
\sin^2(t) = 0 \\
\sin(t) = 0 \\
t = \sin^{-1}(0) + 2k\pi, \pi - \sin^{-1}(0) + 2k\pi \\
t = 0 + 2k\pi, \pi + 2k\pi \\
t = k\pi
\end{gathered}
$$

8. $4\sin^2(x)\tan(x) - \tan(x) = 0, \qquad 0 \le x < 2\pi$

Solution:

$$
\begin{gathered}
\tan(x)(4\sin^2(x) - 1) = 0 \\
\tan(x) = 0, \qquad \sin^2(x) = \frac{1}{4} \\
x = \tan^{-1}(0), \tan^{-1}(0) + \pi, \qquad \sin(x) = \pm\frac{1}{2} \\
x = 0, \pi, \qquad x = \sin^{-1}\left(\frac{1}{2}\right), \pi - \sin^{-1}\left(\frac{1}{2}\right), 2\pi + \sin^{-1}\left(-\frac{1}{2}\right), \pi - \sin^{-1}\left(\frac{1}{2}\right) \\
x = 0, \pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{7\pi}{6}
\end{gathered}
$$

9. $\sin(2\omega) - 3\sin(\omega) = 0, \qquad 0 \le \omega < 2\pi$

Solution:

$$2\sin(\omega)\cos(\omega) - 3\sin(\omega) = 0$$
$$\sin(\omega)(2\cos(\omega) - 3) = 0$$
$$\sin(\omega) = 0, \qquad \cos(\omega) = \frac{3}{2}$$
$$\big(\text{remember, the range of the cosine is } [-1, 1]\big)$$
$$\sin(\omega) = 0$$
$$\omega = \sin^{-1}(0), \pi - \sin^{-1}(0)$$
$$\omega = 0, \pi$$

10. $2\cos^2(x) - \sin(x) = 1, \qquad 0 \le x < 2\pi$

Solution:

$$2(1 - \sin^2(\theta)) - \sin(\theta) = 1$$
$$2 - 2\sin^2(\theta) - \sin(\theta) = 1$$
$$-2\sin^2(\theta) - \sin(\theta) + 1 = 0$$
$$2\sin^2(\theta) + \sin(\theta) - 1 = 0$$
$$(2\sin(\theta) - 1)(\sin(\theta) + 1) = 0$$
$$\sin(\theta) = \frac{1}{2}, \qquad \sin(\theta) = -1$$
$$\theta = \sin^{-1}\left(\frac{1}{2}\right), \pi - \sin^{-1}\left(\frac{1}{2}\right), \sin^{-1}(-1)$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

11. $3\sin^2(\beta) - 2\cos(\beta) = 2, \qquad 0 \le \beta < 2\pi$

Solution:

$$3(1 - \cos^2(\beta) - 2\cos(\beta) = 2$$
$$3 - 3\cos^2(\beta) - 2\cos(\beta) = 2$$
$$-3\cos^2(\beta) - 2\cos(\beta) + 1 = 0$$
$$3\cos^2(\beta) + 2\cos(\beta) - 1 = 0$$
$$(3\cos(\beta) - 1)(\cos(\beta) + 1) = 0$$
$$\cos(\beta) = \frac{1}{3}, \qquad \cos(\beta) = -1$$
$$\beta = \cos^{-1}\left(\frac{1}{3}\right), 2\pi - \cos^{-1}\left(\frac{1}{3}\right), \qquad \cos(\beta) = \cos^{-1}(-1)$$
$$\beta = \cos^{-1}\left(\frac{1}{3}\right), 2\pi - \cos^{-1}\left(\frac{1}{3}\right), \pi$$

12. $2\cos^2(t) + 7\sin(t) = 5, \qquad 0 \le t < 2\pi$

Solution:

$$2(1 - \sin^2(t)) + 7\sin(t) = 5$$
$$2 - 2\sin^2(t)) + 7\sin(t) = 5$$
$$-2\sin^2(t)) + 7\sin(t) - 3 = 0$$
$$2\sin^2(t)) - 7\sin(t) + 3 = 0$$
$$(2\sin(t) - 1)(\sin(t) - 3) = 0$$
$$\sin(t) = \frac{1}{2}, \qquad \sin(t) = 3$$
$$\big(\text{remember, the range of the sine is } [-1, 1]\big)$$
$$t = sin^{-1}\left(\frac{1}{2}\right), \pi - sin^{-1}\left(\frac{1}{2}\right)$$
$$t = \frac{\pi}{6}, \frac{5\pi}{6}$$